N 69 12010

NASA CR 97757

Global Solutions by Analytic Parameter Continuation for a Class of Non-linear Singular Integral Equations*

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1. Introduction

In this paper we shall consider non-linear singular integral equations of the form

$$(1.1)\qquad w(z) = B(z) + \lambda \sum_{r=0}^{r_0} b_r(z) L_r[w,\lambda] + \lambda F[z,w(z),\lambda] + \frac{\lambda}{2\pi i} \int_\Gamma \frac{N[t,z,w(t),\lambda]}{t-z}\, dt ,$$

where B, b_r, F and N are complex-valued functions; L_r are bounded complex-valued functionals; t, z on Γ; the Cauchy principle value is understood by the integral; λ is a real or complex parameter; L_r, F and N are analytic functions (in a sense to be explained) of λ, $\operatorname{Re} w$ and $\operatorname{Im} w$; for λ in some open connected set R_λ containing the origin, $\operatorname{Re} w$ in some neighborhood of $\operatorname{Re} B$ and $\operatorname{Im} w$ in some neighborhood of $\operatorname{Im} B$.

It is the purpose of this paper:

1. to show that if we know the solution of (1.1) for $\lambda = \lambda_0 \in R_\lambda$, then within the class of functions $\{\alpha(z,\lambda)\}$ --the members of which are Hölder continuous in z (fixed exponent) for z on Γ and analytic in λ for λ in some open connected set--there exists a unique member $w(z,\lambda)$ which is a solution of (1.1) on some neighborhood $\mathcal{B}_{\lambda_0}$ of λ_0;

(As an immediate consequence of this, we get the existence of a unique solution of (1.1) provided $|\lambda|$ is sufficiently small.)

2. to show that if in addition to having (i) the analyticity requirements we have (ii) a necessary condition for a function $w(z,\lambda)$ to be a solution is that there exist regions $\mathcal{D}_\lambda$ and $\mathcal{D}_w$ such that if $\lambda \in \mathcal{D}_\lambda$ then $w(z,\lambda) \in \mathcal{D}_w$ for $z \in \Gamma$ (in particular $\mathcal{D}_{\lambda_0} \subset \mathcal{D}_\lambda$), then the solution obtained for $\lambda \in \mathcal{D}_{\lambda_0}$ can be continued analytically to a solution for λ beyond $\mathcal{D}_{\lambda_0}$;

3. to find estimates for how far beyond $\mathcal{D}_{\lambda_0}$ the solution can be continued;

4. to show that, under certain general conditions, the solution can be analytically continued to all of $\mathcal{D}_\lambda$.

The primary methods that have been used for parameter continuation are: (1) the analytic method used in this paper, see also Babaev [2]; (2) the Newton-Kantorovic method, see e.g. Moore [7] and Anselone and Moore [1] and their bibliographies; and (3) topological methods, see e.g. Ficken [3], Krasnoselskii [6], and Pogorzelskii [9] and his bibliography.

Non-linear singular integral equations arise in the non-linear Riemann-Hilbert problem [5] and differential equations. Examples of the class of integral equations considered here, arise in free boundary value problems of hydroelasticity and electroelasticity [4], [10], [11], and [12]. It was these latter problems that led to the investigation of integral equations of the type considered in this paper. Experimentally, it is necessary to get solutions for "large" values of the parameter, since only for large values is it possible to make physical measurements.

It is of interest to note that the suspicion of the result and the motivation for its pursuit came about after Glen Culler kindly put an integral equation of the type considered here, arising in hydroelasticity, on an on-line computer. The computer results indicated, for the physical constants used, the restriction that λ be small, was unnecessary.

2. Preliminary Definitions

Before stating the results we make the following assumptions and definitions.

By Γ, we shall mean a simple closed contour.

DEFINITION 1. Let $H(\nu)$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $0 < \nu_j \leq 1$, be the set of all complex-valued functions $\varphi(t)$, $t = (t_1, t_2, \ldots, t_n)$, defined for complex t_j on Γ, $1 \leq j \leq n$, and for which there exist constants $k_\varphi(\nu_j)$, $1 \leq j \leq n$, such that

$$|\varphi(t) - \varphi(z)| \leq \sum_{j=1}^{n} k_\varphi(\nu_j) |t_j - z_j|^{\nu_j}$$

for all complex t_j and z_j, $1 \leq j \leq n$, on Γ.

DEFINITION22. Let $\varphi(t) \in H(\nu)$ and let

$$\|\varphi(t)\|_\nu = M_\varphi + k_\varphi(\nu) \,, \tag{2.1}$$

where

$$M_\varphi = \max_{\substack{t_j \in \Gamma \\ 1 \leq j \leq n}} |\varphi(t)| \,, \tag{2.2}$$

$$k_\varphi(\nu) = \sup_{\substack{t_j, z_j \in \Gamma \\ 1 \leq j \leq n}} \frac{|\varphi(t) - \varphi(z)|}{\sum_{j=1}^{n} |t_j - z_j|^{\nu_j}} \,. \tag{2.3}$$

Remark 1. It is clear that $\| \|_\nu$ is a norm on $H(\nu)$ and it can easily be shown that $H(\nu)$ equipped with this norm is a Banach space.

Remark 2. In the event $t_j = r_j + is_j$ is replaced by (r_j, s_j) and $z_j = x_j + iy_j$ is replaced by (x_j, y_j) in the above, then $|t_j - z_j|^{\nu_j}$ is replaced by

$$[(r_j - x_j)^2 + (s_j - y_j)^2]^{\nu_j/2}.$$

Let $C^n = C \times C \times \cdots \times C$ (n-times) where C is the complex plane. Then $\Gamma^n = \Gamma \times \cdots \times \Gamma$ (n-times) $\subset C^n$. Let $w(z) = u(x,y) + iv(x,y)$ be defined on Γ and $\mathcal{D}_w$ and $\mathcal{D}_\lambda$ be open subsets of C. Let $0 < \mu \leq 1$ and

$$\mathcal{D}_w(\mu) = \{(u,v) : u + iv \in \mathcal{D}_w \text{ for } (x,y) \in \Gamma \text{ and } u + iv \in H(\mu)\} .$$

DEFINITION 3. Let

$$A[z,u,v,\lambda] : (z,u,v,\lambda) \in \Gamma^n \times \mathcal{D}_w \times \mathcal{D}_\lambda \to C$$

and let $(q_1, q_2, \lambda_0) \in \mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$. We shall say "$A[z,u,v,\lambda]$ is an analytic function at (q_1, q_2, λ_0) with $H[\nu, A^*, a, b, c]$ coefficients" and mean that A can be represented uniquely as

$$A[z,u,v,\lambda] = \sum_{j,k,\ell=0}^{\infty} a_{jk\ell}(z)[u-q_1]^j (v-q_2]^k [\lambda-\lambda_0]^\ell \tag{2.4}$$

provided $|u-q_1|$, $|v-q_2|$ and $|\lambda-\lambda_0|$ are sufficiently small, where $a_{jk\ell}(z)$ are complex-valued functions in $H(\nu)$ for $z_j \in \Gamma$;

$a_{jk\ell}(z)$ depend on q_1, q_2, λ_0; and for some $A^*(q_1,q_2,\lambda_0)$, $a(q_1,q_2,\lambda_0)$, $b(q_1,q_2,\lambda_0)$, $c(q_1,q_2,\lambda_0)$,

$$\|a_{jk\ell}(z)\|_\nu \le A^*(q_1,q_2,\lambda_0)a^j(\ldots)b^k(\ldots)c^\ell(\ldots) \tag{2.5}$$

where A^*, a, b, c are real and independent of z.

Let $\mathcal{D}_\lambda$ be an open subset of C and let

$$\gamma(x,y,\lambda) = a(x,y,\lambda) + ib(x,y,\lambda)$$

be a complex-valued function

$$\gamma(x,y,\lambda) : (x,y,\lambda) \in \Gamma\times\mathcal{D}_\lambda \to C. \tag{2.6}$$

DEFINITION 3A. Let $\lambda_0 \in \mathcal{D}_\lambda$. We shall say "$\gamma(x,y,\lambda)$ is an analytic function at λ_0 with $H[\nu,\lambda_0]$ (ν = one component vector) coefficients" and mean that γ can be represented uniquely as:

$$\gamma(x,y,\lambda) = \sum_{j=0}^{\infty} \frac{1}{j!} c_j(x,y,\lambda_0)(\lambda-\lambda_0)^j$$

provided $|\lambda-\lambda_0|$ is sufficiently small, where $c_j(x,y,\lambda_0)$ are complex-valued functions in $H(\nu)$ for $x+iy \in \Gamma$, and for some $A^*(\lambda_0)$,

$$\|c_j(x,y,\lambda_0)\|_\nu \le A^*(\lambda_0)$$

where $A^*(\lambda_0)$ is a constant independent of (x,y).

DEFINITION 4. Let $\mathcal{D}_w \subset C$ and $\mathcal{D}_\lambda \subset C$ be open arc-wise connected sets and let $A[z,u,v,\lambda]$ be defined for $(z,u,v,\lambda) \in \Gamma^n \times \mathcal{D}_w \times \mathcal{D}_\lambda$. We shall say "$A[z,u,v,\lambda]$ is an analytic function on $\mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$ with $H[\nu,A^*,a,b,c,q_1,q_2,\lambda_0]$ coefficients" and mean that A is analytic at each point $(q_1,q_2,\lambda_0) \in \mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$ with $H[\nu,A^*,a,b,c]$ coefficients.

DEFINITION 4A. It is clear what is meant by "$\gamma(x,y,\lambda)$ is an analytic function on $\mathcal{D}_\lambda$ with $H[\nu,\lambda]$ coefficients".

Remark. If $\gamma(x,y,\lambda)$ is an analytic function at λ_0 with $H[\nu,\lambda]$ coefficients, it is clear that it is analytic on some neighborhood $\mathcal{D}_\lambda$ of λ_0 with $H[\nu,\lambda]$ $(\lambda \in \mathcal{D}_\lambda)$ coefficients (if λ complex it is clear, if λ real then extend γ to be an analytic function of complex variable).

DEFINITION 5. We shall say "$A[z,u,v,\lambda]$ is a uniformly analytic function on $\mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$ with $H(\nu,A^*,a,b,c)$ coefficients" and mean that $A[z,u,v,\lambda]$ is analytic on $\mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$ with $H[\nu,A^*,a,b,c,q_1,q_2,\lambda_0]$ coefficients, and in addition, there exist constants R_u, R_v, R_λ such that for all $(q_1,q_2,\lambda_0) \in \mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$ and $|u-q_1| < R_u$, $|v-q_2| < R_v$, $|\lambda-\lambda_0| < R_\lambda$, and for all $a_{jk\ell}(z)$ of (2.4) we have:

$$\|a_{jk\ell}(z)\|_\nu \le A^* a^j b^k c^\ell \tag{2.8}$$

where A^*, a, b, c are independent of z and all points of $\mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$.

DEFINITION 5A. It is clear what is meant by "$\gamma(x,y,\lambda)$ is a uniformly analytic function on $\mathcal{D}_\lambda$ with $H[\nu]$ coefficients".

Remark. If $\gamma(x,y,\lambda)$ is an analytic function at λ_0 with $H[\nu]$ coefficients, then $\gamma(x,y,\lambda)$ is a uniformly analytic function on some neighborhood of λ_0 with $H[\nu]$ coefficients.

THEOREM 1. Given the non-linear singular integral equation with real or complex parameter λ:

$$\text{(E)} \qquad w(z) = B(z) + \sum_{r=0}^{r_0} b_r(z) L\{P^r[\sigma,u(\xi,\eta),v(\xi,\eta),\lambda]\}$$
$$+ F[z,u(x,y),v(x,y),\lambda] + \frac{1}{2\pi i}\int_\Gamma \frac{N[\sigma,z,u(\xi,\eta),v(\xi,\eta),\lambda]}{\sigma - z}\,d\sigma$$

where $z = x + iy \in \Gamma$, $\sigma = \xi + i\eta \in \Gamma$, $w = u + iv$, and by the integral, we mean the Cauchy-Principal value. Assume:

(H.1) $B(z)$ and $b_r(z)$ are single-valued functions of z which are in $H(\mu)$, $0 < \mu < 1$, for z on Γ, $\|b_r(z)\|_\mu \le b^0$;

(H.2) L is a complex-valued linear bounded functional defined on the set of complex-valued functions continuous on Γ, with $L^* = \|L\|$; **

(H.3) $w_0(z) = u_0(x,y) + iv_0(x,y) \in H(\mu)$;

(H.4) $F[z,u,v,\lambda]$ and $P^r[\cdots]$ are defined and single-valued on $\Gamma \times \mathcal{D}_w \times \mathcal{D}_\lambda$, and $N[\cdots]$ is defined and single-valued on $\Gamma \times \Gamma \times \mathcal{D}_w \times \mathcal{D}_\lambda$ (where $\mathcal{D}_w$ is a neighborhood of $\{w_0(z) : z \in \Gamma\}$ and $\mathcal{D}_\lambda$ is a neighborhood of λ_0) and they are analytic functions on $\mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$ with

$H[\mu, p_0, a_p, b_p, c_p, q_1, q_2, \lambda_0]$ coefficients for P^r,

$H[\mu, f_0, a_f, b_f, c_f, q_1, q_2, \lambda_0]$ coefficients for F and

$H[\mu, \mu_1, a_n, b_n, c_n, q_1, q_2, \lambda_0]$ coefficients for N, $0 < \mu < \mu_1 < 1$;

(H.5) $w_0(z)$ is a solution of (E) for $\lambda = \lambda_0$.

Conclusion:

(C.1) There exists a unique solution $w(z,\lambda)$ of (E) with $w(z,\lambda_0) = w_0(z)$ which is an analytic function of λ on $\mathcal{B}_{\lambda_0}$ with $H(\mu,\lambda)$ coefficients where $\mathcal{B}_{\lambda_0}$ is the circle with center at $\lambda_0 - ec$ and radius $\sqrt{e^2c^2 + e}$ where

$$e = [(2d^{-1}U^*)^2 - c^2]^{-1},$$

with

$$d = [2(a+b)]^{-1},$$

$$a = \max\{a_p(u_0,v_0,\lambda_0), a_f(\ldots), a_n(\ldots)\},$$

$$b = \max\{b_p(u_0,v_0,\lambda_0), b_f(\ldots), b_n(\ldots)\},$$

$$c = \max\{c_p(u_0,v_0,\lambda_0), c_p(\ldots), c_n(\ldots)\},$$

and

$$U^* = c[L^* r_0 b^0 p_0 + f_0 + (2\pi)^{-1}K^*(\mu)n_0]$$

(see H.4 and footnote to H.2) where $K^*(\mu)$ is the constant found in the corollary to Lemma 2 below (viz. $K^*(\mu) = \max\{\pi, C'(\mu) + C(\mu,\mu)\}$, $C(\mu,\mu)$ = positive constant depending on Γ and μ, $C'(\mu) = \sup_{t\in\Gamma}\int_\Gamma |\tau - t|^{\mu-1}d\tau$).

(C.2) *If* λ *is real then the solution is valid for*

$$\lambda_0 - \frac{d}{2U^* - dc} < \lambda < \lambda_0 + \frac{d}{2U^* + dc} \, .$$

(C.3)

$$\|w(z,\lambda) - w_0(z)\|_\mu \le d - \sqrt{d^2 - t(|\lambda - \lambda_0|)}$$

where

$$t(\tilde\lambda) = U^*[a+b]^{-1} \frac{\tilde\lambda}{1 - c\tilde\lambda} = 2dU^* \frac{\tilde\lambda}{1 - c\tilde\lambda} \, .$$

Proof: Before proving the theorem, we shall state and prove several lemmas which are needed in the proof.

LEMMA 1. *Let* $\varphi(\tau,\zeta)$ *be a complex single-valued function of the two complex variables* τ *and* ζ *for* $\tau \in \Gamma$, $\zeta \in \mathcal{D}$, ($\mathcal{D}$ *some region or line of the complex plane) and let*

1. $|\varphi(\tau,\zeta)| < M_\varphi$, $\quad \tau \in \Gamma$, $\quad \zeta \in \mathcal{D}$,

and

2. $|\varphi(\tau_2,\zeta_2) - \varphi(\tau_1,\zeta_1)| < k_\varphi(\mu,\nu)[|\tau_2 - \tau_1|^\mu + |\zeta_2 - \zeta_1|^\nu]$,

$$\tau_i \in \Gamma, \quad \zeta_i \in \mathcal{D}, \quad i = 1,2,$$

$0 < \mu < 1$, $\quad 0 < \nu \le 1$. *Then the singular integral*

$$\Phi(t,\zeta) = \int_\Gamma \frac{\varphi(\tau,\zeta)}{\tau - t} \, d\tau$$

satisfies

1. $|\Phi(t,\zeta)| < \pi M_\varphi + C'(\mu)k_\varphi(\mu,\nu)$,

and

2. $|\Phi(t_2,\zeta_2) - \Phi(t_1,\zeta_1)| < C(\mu,\nu^*)(k_\varphi(\mu,\nu)[|t_2-t_1|^\mu + |\zeta_2-\zeta_1|^{\nu^*}]$

where

$\nu^* < \nu$ is an arbitrary positive constant less than ν,

$C(\mu,\nu^*)$ = positive constant depending on Γ, μ, and the choice of ν^*,

$C'(\mu) = \sup_{t\in\Gamma} \int_\Gamma |\tau-t|^{\mu-1} d\tau$.

Proof: See Pogorzelski [4].

As a consequence of this, we have

LEMMA 2. Let $\varphi(\tau,\sigma) \in H(\mu,\nu)$, $0<\mu<\nu<1$, and

$$\Phi(t,\sigma) = \int_\Gamma \frac{\varphi(\tau,\sigma)}{\tau - t}\, d\tau \ .$$

Then

1. $\Phi(t,\sigma) \in H(\mu,\mu)$

and

2. $\|\Phi(t,\sigma)\|_{(\mu,\mu)} \leq K(\mu)\|\varphi(\tau,\sigma)\|_{(\mu,\nu)}$

where

$$K(\mu) = \max\{\pi, C'(\mu) + C(\mu,\mu)\}$$

with $C(\mu,\mu)$ and $C'(\mu)$ given in Lemma 1.

Proof: The fact that $\Phi(t,\sigma) \in H(\mu,\mu)$ follows immediately from the fact that $\Phi(t,\sigma) \in H(\mu,\nu^*)$ for any $\nu^* < \nu$ by Lemma 1 and $\mu < \nu$ by assumption.

Next, in the notation of Definition 2, let

$$\|\varphi(\tau,\sigma)\|_{(\mu,\nu)} = M_\varphi + k_\varphi(\mu,\nu) .$$

Then by Lemma 1, since $\mu < \nu$ by assumption, we can choose $\nu^* = \mu$ and get

$$|\Phi(t,\sigma)| \leq \pi M_\varphi + C'(\mu)k_\varphi(\mu,\nu)$$

and

$$|\Phi(t_2,\sigma_2) - \Phi(t_1,\sigma_1)| < C(\mu,\mu)k_\varphi(\mu,\nu)[|t_2-t_1|^\mu + |\sigma_2-\sigma_1|^\mu] .$$

Thus

$$\|\Phi(t,\sigma)\|_{(\mu,\mu)} < \pi M_\varphi + [C'(\mu) + C(\mu,\mu)]k_\varphi(\mu,\nu)$$
$$\leq K(\mu)\|\varphi(\tau,\sigma)\|_{(\mu,\nu)}$$

where

$$K(\mu) = \max\{\pi, C'(\mu) + C(\mu,\mu)\} ,$$

which proves the lemma.

As a consequence, we have the

COROLLARY: Let $\varphi(\tau,t) \in H(\mu,\nu)$, $0 < \mu < \nu < 1$ and let

$$\Phi(t) = \int_\Gamma \frac{\varphi(\tau,t)}{\tau - t}\, d\tau \ .$$

Then

1. $\Phi(t) \in H(\mu)$

and

2. $\|\Phi(t)\|_\mu \le K^*(\mu)\|\varphi(\tau,t)\|_{(\mu,\nu)}$

where

$$K^*(\mu) = \max\{\pi, C'(\mu) + 2C(\mu,\mu)\} \ .$$

Proof: The fact that $\Phi(t) \in H(\mu)$ follows from 1 of Lemma 2. To obtain 2 we specialize the proof of Lemma 2 to $\sigma = t$.

Next we shall need

LEMMA 3. If $\varphi(t,z) \in H(\mu,\nu)$ and $\psi(t) \in H(\mu)$, $(\psi_0(z) \in H(\nu))$, then

1. $\varphi(t,z)\psi(t) \in H(\mu,\nu)$, $\quad (\varphi(t,z)\psi_0(z) \in H(\mu,\nu))$,

and

2. $\|\varphi(t,z)\psi(t)\|_{(\mu,\nu)} \le \|\varphi(t,z)\|_{(\mu,\nu)}\|\psi(t)\|_\mu$,

   $(\|\varphi(t,z)\psi_0(z)\|_{(\mu,\nu)} \le \|\varphi(t,z)\|_{(\mu,\nu)}\|\psi_0(z)\|_\nu)$.

Proof: It is clear that $\varphi(t,z)\psi(t) \in H(\mu,\nu)$.
In order to prove 2, let

$$\|\varphi\|_{(\mu,\nu)} = M_\varphi + k_\varphi(\mu,\nu) , \qquad \|\psi\|_\mu = M_\psi + k_\psi(\mu) ,$$

(2.9)

$$\|\varphi\psi\|_{(\mu,\nu)} = M_{\varphi\psi} + k_{\varphi\psi}(\mu,\nu) ,$$

where

$$M_\varphi = \max_{t,z\in\Gamma} |\varphi(t,z)| , \qquad M_\psi = \max_{t\in\Gamma} |\psi(t))| ,$$

(2.10)

$$M_{\varphi\psi} = \max_{t,z\in\Gamma} |\varphi(t,z)\psi(t)| ,$$

$$\text{(2.11)} \qquad k_\varphi(\mu,\nu) = \sup_{\substack{t_j,z_j\in\Gamma \\ j=1,2}} \frac{|\varphi(t_2,z_2) - \varphi(t_1,z_1)|}{|t_2-t_1|^\mu + |z_2-z_1|^\nu} ,$$

$$\text{(2.12)} \qquad k_\psi(\mu) = \sup_{\substack{t_j\in\Gamma \\ j=1,2}} \frac{|\psi(t_2) - \psi(t_1)|}{|t_2-t_1|^\mu} , \quad \text{and}$$

$$\text{(2.13)} \qquad k_{\varphi\psi}(\mu,\nu) = \sup_{\substack{t_j,z_j\in\Gamma \\ j=1,2}} \frac{|\varphi(t_2,z_2)\psi(t_2) - \varphi(t_1,z_1)\psi(t_1)|}{|t_2-t_1|^\mu + |z_2-z_1|^\nu} .$$

From (2.10), we see that

$$\text{(2.14)} \qquad M_{\varphi\psi} \le M_\varphi M_\psi .$$

Next note that for $t_j, z_j \in \Gamma$, $j = 1,2$,

$$
\begin{aligned}
&|\varphi(t_2,z_2)\psi(t_2) - \varphi(t_1,z_1)\psi(t_1)| \\
&\qquad \le |\varphi(t_2,z_2)|\ |\psi(t_2) - \psi(t_1)| + |\psi(t_1)|\ |\varphi(t_2,z_2) - \varphi(t_1,z_1)| \\
&\qquad \le M_\varphi k_\psi(\mu)|t_2 - t_1|^\mu + M_\psi k_\varphi(\mu,\nu)[|t_2 - t_1|^\mu + |z_2 - z_1|^\nu] \\
&\qquad \le [M_\varphi k_\psi(\mu) + M_\psi k_\varphi(\mu,\nu)][|t_2 - t_1|^\mu + |z_2 - z_1|^\nu] .
\end{aligned}
$$

Thus

$$
\begin{aligned}
M_{\varphi\psi} + k_{\varphi\psi}(\mu,\nu) &\le M_\varphi M_\psi + M_\varphi k_\psi(\mu) + M_\psi k_\varphi(\mu,\nu) \\
&\le [M_\varphi + k_\varphi(\mu,\nu)][M_\psi + k_\psi(\mu)]
\end{aligned}
$$

which is the result. The proof for $\varphi(t,z)\psi_0(z)$ follows in exactly the same way.

Now we are in a position to prove Theorem 1. The proof is by the method of majorants.

By assumption (H.4), we have

$$
(2.15.1) \qquad P^r[\sigma,u,v,\lambda] = \sum_{j,k,\ell=0}^{\infty} p^r_{jk\ell}(\sigma)[u - u_0(\xi,\eta)]^j[v - v_0(\xi,\eta)]^k[\lambda-\lambda_0]^\ell,
$$

$$
(2.15.2) \qquad F[z,u,v,\lambda] = \sum_{j,k,\ell=0}^{\infty} f_{jk\ell}(z)[u - u_0(x,y)]^j[v - v_0(x,y)]^k[\lambda-\lambda_0]^\ell,
$$

$$
(2.15.3) \qquad N[\sigma,z,u,v,\lambda] = \sum_{j,k,\ell=0}^{\infty} n_{jk\ell}(\sigma,z)[u - u_0(\xi,\eta)]^j[v - v_0(\xi,\eta)]^k[\lambda-\lambda_0]^\ell,
$$

where $p^r_{jk\ell}(\sigma)$, $f_{jk\ell}(z)$ and $n_{jk\ell}(\sigma,z)$ are complex-valued functions with p, $f \in H(\mu)$ and $n \in H(\mu,\mu_1)$, $0 < \mu < \mu_1 < 1$ for $\sigma, z \in \Gamma$. Also for $\sigma, z \in \Gamma$:

$$\|p^r_{jk\ell}(\sigma)\|_\mu \le p_0 a_p^j b_p^k c_p^\ell \,, \tag{2.16.1}$$

$$\|f_{jk\ell}(z)\|_\mu \le f_0 a_f^j b_f^k c_f^\ell \,, \tag{2.16.2}$$

$$\|n_{jk\ell}(\sigma,z)\|_{(\mu,\mu_1)} \le n_0 a_n^j b_n^k c_n^\ell \,, \tag{2.16.3}$$

$$\|b_r(z)\|_\mu \le b^0 \,, \tag{2.16.4}$$

where p_0, f_0, n_0, a_p, b_p, $c_p, \ldots, c_n$ are constants independent of z and σ (these depend on u_0, v_0, and λ_0, but since u_0, v_0, λ_0 will be held fixed, we omit mention of them), and b^0 is a constant.

We now introduce the following notation:

$$P^r_m[\sigma,u,v,\lambda] = \sum_{\ell=m}^{\infty} \tilde{p}^r_\ell[\sigma,u,v]\tilde{\lambda}^\ell \,, \qquad m = 0,1,2,\ldots, \tag{2.17.1}$$

where

$$\tilde{p}^r_\ell[\sigma,u,v] = \sum_{j,k=0}^{\infty} p^r_{jk\ell}(\sigma)[u - u_0(\xi,\eta)]^j[v - v_0(\xi,\eta)]^k \tag{2.17.2}$$

and

$$\tilde{\lambda} = \lambda - \lambda_0 \,;$$

$$F_m[z,u,v,\lambda] = \sum_{\ell=m}^{\infty} \tilde{f}_\ell[z,u,v]\tilde{\lambda}^\ell, \qquad m = 0,1,2,\ldots, \tag{2.18.1}$$

where

$$\tilde{f}_\ell[z,u,v] = \sum_{j,k=0}^{\infty} f_{jk\ell}(z)[u - u_0(x,y)]^j[v - v_0(x,y)]^k \; ; \tag{2.18.2}$$

$$N_m[\sigma,z,u,v,\lambda] = \sum_{\ell=m}^{\infty} \tilde{n}_\ell[\sigma,z,u,v]\tilde{\lambda}^\ell , \tag{2.19.1}$$

where

$$\tilde{n}_\ell[\sigma,zu,v] = \sum_{j,k=0}^{\infty} n_{jk\ell}(\sigma,z)[u - u_0(\xi,\eta)]^j[v - v_0(\xi,\eta)]^k . \tag{2.19.2}$$

In this notation we have:

$$\begin{aligned} L\{P^r[\sigma,u(\xi,\eta),v(\xi,\eta),\lambda]\} = {} & L\{\tilde{P}_0^r[\sigma,u(\xi,\eta),v(\xi,\eta)]\} \\ & + L\{P_1^r[\sigma,u(\xi,\eta),v(\xi,\eta),\lambda]\} , \end{aligned} \tag{2.20.1}$$

$$\begin{aligned} F[z,u(x,y),v(x,y),\lambda] = {} & \tilde{f}_0[z,u(x,y),v(x,y)] \\ & + F_1(z,u(x,y),v(x,y),\lambda] \end{aligned} \tag{2.20.2}$$

$$\begin{aligned} N[\sigma,z,u(\xi,\eta),v(\xi,\eta),\lambda] = {} & \tilde{n}_0[\sigma,z,\ldots \qquad] \\ & + N_1[\ldots \qquad] . \end{aligned} \tag{2.20.3}$$

Thus (E) can be written:

$$\text{(E)} \qquad w(z) = B(z) + \sum_{r=0}^{r_0} b_r(z) L\{\tilde{P}_0^r[\sigma, u(\xi,\eta), v(\xi,\eta)]\}$$

$$+ \tilde{F}_0[z, u(x,y), v(x,y)] + \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{N}_0[\sigma, z, u(\xi,\eta), v(\xi,\eta)]}{\sigma - z}\, d\sigma$$

$$+ \sum_{r=0}^{r_0} b_r(z) L\{P_1^r[\sigma, u(\xi,\eta), v(\xi,\eta), \lambda]\} + F_1[z, u(x,y), v(x,y), \lambda]$$

$$+ \frac{1}{2\pi i}\int_\Gamma \frac{N_1[\sigma, z, u(\xi,\eta), v(\xi,\eta), \lambda]}{\sigma - z}\, d\sigma .$$

By (H.5) $w_0(z)$ is a solution of (E) for $\lambda = \lambda_0$, i.e., $\tilde{\lambda} = 0$. Thus (E) can be written as:

$$\text{(E)} \qquad w(z) = w_0(z) + \sum_{r=0}^{r_0} b_r(z) L\{P_1^r[\sigma, u(\xi,\eta), v(\xi,\eta)\lambda]\}$$

$$+ F_1[z, u(x,y), v(x,y), \lambda]$$

$$+ \frac{1}{2\pi i}\int_\Gamma \frac{N_1[\sigma, z, u(\xi,\eta), v(\xi,\eta), \lambda]}{\sigma - z}\, d\sigma .$$

We now try to find a solution of the form:

$$\text{(2.21)} \qquad w(z,\lambda) = u(x,y,\lambda) + iv(x,y,\lambda)$$

(u and v not necessarily real), where

$$(2.22) \qquad u(x,y,\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} u_k(x,y)\tilde{\lambda}^k, \qquad u_k \text{ real and } \in H(\mu),$$

$$(2.23) \qquad v(x,y,\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} v_k(x,y)\tilde{\lambda}^k, \qquad v_k \text{ real and } \in H(\mu).$$

Since $P_1^r[\ldots,\lambda_0] = F_1[\ldots,\lambda_0] = N_1[\ldots,\lambda_0] = 0$ we see that if $w(z,\lambda)$ is to be a solution, we must have:

$$u_k(x,y) = \frac{\partial^k}{\partial\lambda^k} u(x,y,\lambda_0), \qquad v_k(x,y) = \frac{\partial^k}{\partial\lambda^k} v(x,y,\lambda_0) \;,$$

$k = 1,2,3,\ldots$ and

$$(2.24) \qquad u_0(x,y) + iv_0(x,y) = u(x,y,\lambda_0) + iv(x,y,\lambda_0)$$
$$= w_0(z) \;.$$

Also

$$(2.25) \qquad u_1(x,y) + iv_1(x,y) = u_\lambda(x,y,\lambda_0) + iv_\lambda(x,y,\lambda_0)$$
$$= \sum_{r=0}^{r_0} b_r(z)L\{\tilde{P}_1^r[\sigma,u_0,v_0]\} + \tilde{F}_1[z,u_0,v_0]$$
$$+ \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{N}_1[\sigma,z,u_0,v_0]}{\sigma - z}\, d\sigma$$

$$= \sum_{r=0}^{r_0} b_r(z)L\{p_{001}^r(\sigma)\} + f_{001}(z) + \frac{1}{2\pi i}\int_\Gamma \frac{n_{001}(\sigma,z)}{\sigma - z}\, d\sigma \;,$$

(2.26)

$$u_2(x,y) + iv_2(x,y) = u_{\lambda\lambda}(x,y,\lambda_0) + iv_{\lambda\lambda}(x,y,\lambda_0)$$

$$= 2\sum_{r=0}^{r_0} b_r(z)L\{\tilde{p}_2^r[\sigma,u_0,v_0] + \frac{\partial}{\partial u}\tilde{p}_1^r[\sigma,u_0,v_0]u_1(\xi,\eta) + \frac{\partial}{\partial v}\tilde{p}_1^r[\ldots]v_1(\xi,\eta)\}$$

$$+ 2\{\tilde{f}_2[z,u_0,v_0] + \frac{\partial}{\partial u}\tilde{f}_1[\ldots]u_1(x,y) + \frac{\partial}{\partial v}\tilde{f}_1[\ldots]v_1(x,y)\}$$

$$+ \frac{1}{\pi i}\int_\Gamma \frac{\{\tilde{n}_2[\sigma,z,u_0,v_0] + \frac{\partial}{\partial u}\tilde{n}_1[\ldots]u_1(\sigma) + \frac{\partial}{\partial v}\tilde{n}_1[\ldots]v_1(\sigma)\}}{\sigma - z}\,d\sigma$$

$$= 2\sum_{r=0}^{r_0} b_r(z)L\{p_{002}^r(\sigma) + p_{101}(\sigma)u_1(\xi,\eta) + p_{011}(\sigma)v_1(\xi,\eta)\}$$

$$+ 2\{f_{002}(z) + f_{101}(z)u_1(x,y) + f_{011}(z)v_1(x,y)\}$$

$$+ \frac{1}{\pi i}\int_\Gamma \frac{\{n_{002}(\sigma,z) + n_{101}(\sigma,z)u_1(\xi,\eta) + n_{011}(\sigma,z)v_1(\xi,\eta)\}}{\sigma - z}\,d\sigma$$

and in general $u_{n+1}(x,y) + iv_{n+1}(x,y)$ is given in terms of $u_j(x,y)$, $v_j(x,y)$, $1 \leq j \leq n$, and given functions viz. $p_{jk\ell}^r$, $f_{jk\ell}$ and $n_{jk\ell}$ where $0 \leq j+k+\ell \leq n+1$. Thus if (2.21) is to give a solution of (E), u_n and v_n must be determined in the above way. Moreover, if (2.22) and (2.23) are convergent series, then u_n and v_n, determined in the above fashion, will give a solution of (E) for λ in a neighborhood $\mathcal{D}_\lambda$ of λ_0. Moreover, the solution will be unique within the set of analytic functions on this $\mathcal{D}_\lambda$ with $H(\mu)$ coefficients since (2.21) is a general representative of such functions. Thus the proof of the theorem rests on showing that (2.22) and (2.23) are convergent series when u_n and v_n are determined as indicated above.

To prove convergence, we make the following observations:

$$(2.27)\qquad u_{n+1}(x,y) + iv_{n+1}(x,y) = \sum_{r=0}^{r_0} b_r(z)L\{\mathcal{P}_n[P_1^r,u(\xi,\eta),v(\dots)]\} + \mathcal{F}_n[F_1,u(x,y),v(\dots)] + \frac{1}{2\pi i}\int_\Gamma \frac{\mathcal{N}_n[N_1,u(\xi,\eta),v(\dots)]}{\sigma - z}\,d\sigma$$

for $n \geq 0$, where

$$(2.28)\qquad \mathcal{P}_n[P_1^r,u(\xi,\eta),v(\xi,\eta)] = \sum_{\substack{1\leq j+k+s\leq n+1 \\ s\geq 1}} p_{jks}^r(\sigma)\alpha_s^{(j,k)}[u_1,u_2,\dots,u_n,v_1,\dots,v_n]$$

for $n \geq 1$; $\alpha_s^{(j,k)}$ is a polynomial of degree $\leq n+1-s$ in its arguments with positive rational coefficients. Similarly for the cases when P_1^r is replaced by F_1 and N_1, then $p_{jks}^r(\sigma)$ is replaced by $f_{jks}(z)$ and $n_{jks}(\sigma,z)$ respectively.

With the idea in mind of showing that (2.22) and (2.23), with u_{n+1} and v_{n+1} determined as indicated, are convergent, we consider the auxiliary equation

$$(2.29)\qquad \alpha(x,y) + i\beta(x,y) = \psi_0(z) + L^*b^0\sum_{r=0}^{r_0}\mathcal{P}_1^r[\alpha(x,y),\alpha(x,y),\lambda] + \mathcal{F}_1[\alpha(x,y),\alpha(x,y),\lambda] + (2\pi)^{-1}(K^*(\mu)\mathcal{N}_1[\alpha(x,y),\alpha(x,y),\lambda]$$

(note that the right-hand side involves only α's, no β's), where

$$(2.30)\qquad \mathcal{P}_1^r[\alpha,\beta,\lambda] = p_0 \sum_{\substack{j,k=0\\ \ell=1}}^{\infty} a_p^j b_p^k c_p^\ell [\alpha - u_0(x,y)]^j [\beta - u_0(x,y)]^k [\lambda - \lambda_0]^\ell ,$$

$$(2.31)\qquad \mathcal{F}_1[\alpha,\beta,\lambda] = f_0 \sum_{\substack{j,k=0\\ \ell=1}}^{\infty} a_f^j b_f^k c_f^\ell [\qquad]^j [\qquad]^k [\quad]^\ell ,$$

and

$$(2.32)\qquad \mathcal{N}_1[\alpha,\beta,\lambda] = n_0 \sum_{\substack{j,k=0\\ \ell=1}}^{\infty} a_n^j b_n^k c_n^\ell [\qquad]^j [\qquad]^k [\quad]^\ell ,$$

where the notation in (2.16.1), (2.16.2) and (2.16.3) has been used and $K^*(\mu)$ is the same as in the corollary to Lemma 2.

Next we try to find a solution of (2.29) which is an analytic function on some neighborhood of λ_0 with $H[\mu,\lambda]$ coefficients, i.e., we seek a solution

$$\alpha(x,y,\lambda) + i\beta(x,y,\lambda)$$

for which

$$(2.33)\qquad \alpha(x,y,\lambda) = \sum_{m=0}^{\infty} \frac{1}{m!}\, \alpha_m(x,y)(\lambda - \lambda_0)^m$$

and

$$(2.34)\qquad \beta(x,y,\lambda) = \sum_{m=0}^{\infty} \frac{1}{m!}\, \beta_m(x,y)(\lambda - \lambda_0)^m$$

with $\alpha_m(x,y)$ and $\beta_m(x,y)$ real.

Proceeding as before, we find

$$\alpha_0(x,y) + i\beta_0(x,y) = \alpha(x,y,\lambda_0) + i\beta(x,y,\lambda_0) = w_0(z) = u_0(x,y) + iv_0(x,y) \tag{2.35.0}$$

and

$$\alpha_1(x,y) + i\beta_1(x,y) = \alpha_\lambda(x,y,\lambda_0) + i\beta_\lambda(x,y,\lambda_0) = L^* r_0 b^0 p_0 c_p + f_0 c_f + (2\pi)^{-1} K^*(\mu) n_0 c_n \ . \tag{2.35.1}$$

Thus $\beta_1(x,y) = 0$. In general we get:

$$\begin{aligned} &\alpha_{n+1}(x,y) = \text{positive constant} \ , \\ &\beta_{n+1}(x,y) = 0 \ , \qquad\qquad n \geq 0. \end{aligned} \tag{2.35.n+1}$$

More specifically:

$$\alpha_{n+1} = L^* b^0 \sum_{r=0}^{r_0} \mathcal{B}_n[\mathcal{P}_1^r,\alpha,\alpha] + \mathcal{B}_n[\mathcal{F}_1,\alpha,\alpha] + (2\pi)^{-1} K^*(\mu)\mathcal{B}_n[\eta_1,\alpha,\beta] \ , \tag{2.36}$$

$n \geq 1$, where $\mathcal{B}_n$ is given in (2.28).

Next, note that for $n \geq 1$,

$$\|\mathcal{B}_n[\mathcal{P}_1^r,\alpha,\alpha]\|_\mu = \mathcal{B}_n[\mathcal{P}_1^r,\alpha,\alpha]$$

since all of the $\alpha_m(x,y)$ are constant for $m > 0$, the coefficients a_p, b_p, c_p are constant, and thus $\mathcal{B}_n[\mathcal{P}_1^r,\alpha,\alpha]$ is the positive sum of positive constants. Thus

$$(2.37) \qquad \|\mathcal{B}_n[\rho_1^r,\alpha,\alpha]\|_\mu = \sum_{\substack{1\le j+k+s\le n+1\\ s\ge 1}} a_p^j b_p^k c_p^s \alpha_s^{(j,k)}[\alpha_1,\ldots,\alpha_n,\alpha_1,\ldots,\alpha_n] .$$

A similar result holds when we replace ρ_1^r by $\mathcal{F}_1$ and $\mathcal{N}_1$ respectively. Thus collecting terms, we get that

$$(2.38) \qquad \alpha_{n+1} = \sum_{\substack{1\le j+k+s\le n+1\\ s\ge 1}} U_{jks}\alpha_s^{(j,k)}[\alpha_1,\alpha_2,\ldots,\alpha_n,\alpha_1,\ldots,\alpha_n]$$

where

$$(2.39) \qquad U_{jks} = L^* r_0 p^0 p_0 a_p^j b_p^k c_p^s + f_0 a_f^j b_f^k c_f^s + (2\pi)^{-1} K^*(\mu) n_0 a_n^j b_n^k c_n^s .$$

We now return to (2.27) and note that for $n \ge 0$:

$$(2.40) \qquad \|u_{n+1}(x,y) + iv_{n+1}(x,y)\|_\mu \le b^0 \sum_{r=0}^{r_0} \|L\{\mathcal{B}_n[P^r,u(\xi,\eta),v(\ldots)]\}\|_\mu$$
$$+ \|\mathcal{B}_n[F,u(x,y),v(\ldots)]\|_\mu$$
$$+ (2\pi)^{-1} \left\|\int_\Gamma \frac{\mathcal{B}_n[N,u(\xi,\eta),v(\ldots)]}{\sigma - z}\, d\sigma\right\|_\mu .$$

But

$$\|L\{\mathcal{B}_n[P^r, u(\xi,\eta), v(\xi,\eta)]\}\|_\mu = |L\{\mathcal{B}_n[P^r, u(\xi,\eta), v(\xi,\eta)]\}|$$

$$\leq \sum_{\substack{1\leq j+k+s\leq n+1\\ s\geq 1}} |L\{p^r_{jks}(\sigma)\alpha_s^{(j,k)}[u_1,u_2,\ldots,u_n,v_1,\ldots,v_n]|$$

$$\leq L^* p_0 \sum a_p^j b_p^k c_p^s \max_{\sigma\in\Gamma} |\alpha_s^{(j,k)}[u_1,\ldots,v_n]|$$

$$\leq L^* p_0 \sum a_p^j b_p^k c_p^s \alpha_s^{(j,k)}[\|u_1\|_\mu, \|u_2\|_\mu, \ldots, \|v_n\|_\mu]$$

where we have used (2.16.1) and the fact that the $\alpha_s^{(j,k)}$ are polynomials with positive coefficients.

In a similar way, we get

$$\|\mathcal{B}_n[F, u(x,y), v(x,y)]\|_\mu \leq f_0 \sum a_f^j b_f^k c_f^s \alpha_s^{(j,k)}[\|u_1\|_\mu, \ldots, \|v_{n+1-s}\|_\mu] .$$

Finally we get

$$\left\| \int_\Gamma \frac{\mathcal{B}_n[N, u(\xi,\eta), v(\xi,\eta)]}{\sigma - z}\, d\sigma \right\|_\mu$$

$$\leq \sum_{\substack{1\leq j+k+s\leq n+1\\ s\geq 1}} \left\| \int_\Gamma \frac{n_{jks}(\sigma,z)\alpha_s^{(j,k)}[u_1(\xi,\eta),\ldots,u_n,v_1,\ldots,v_n]}{\sigma - z}\, d\sigma \right\|_\mu$$

$$\leq \sum K^*(\mu)\ \|n_{jks}(\sigma,z)\alpha_s^{(j,k)}[u_1(\xi,\eta),\ldots,v_n]\|_{(\mu,\mu_1)}$$

(where we have used the corollary to Lemma 2);

$$\leq K^*(\mu) \sum \|n_{jks}(\sigma,z)\|_{(\mu,\mu_1)} \alpha_s^{(j,k)}[\|u_1(\xi,\eta)\|_\mu, \ldots, \|v_n(\xi,\eta)\|_\mu]$$

(by Lemma 3);

$$\leq K^*(\mu)n_0 \sum a_n^j b_n^k c_n^s \alpha_s^{(j,k)}[\|\mu_1\|_\mu, \ldots, \|v_n\|_\mu]$$

(by (2.16.3)).

Collecting the above results and combining with (2.27) and (2.39) gives

$$(2.41) \qquad \|u_{n+1}(x,y) + iv_{n+1}(x,y)\|_\mu \leq \sum_{\substack{1\leq j+k+s\leq n+1 \\ s\geq 1}} U_{jks}\alpha_s^{(j,k)}[\|u_1\|_\mu, \ldots, \|v_n\|_\mu] .$$

We next show that the series for $\alpha(x,y,\lambda) + i\beta(x,y,\lambda)$ dominates the series for $u(x,y,\lambda) + iv(x,y,\lambda)$ viz.:

$$(2.42) \qquad \sum_{m=0}^{\infty} \frac{1}{m!} [\alpha_m(x,y) + i\beta_m(x,y)](\lambda-\lambda_0)^m \gg \sum_{m=0}^{\infty} \frac{1}{m!} [u_m(x,y) + iv_m(x,y)](\lambda-\lambda_0)^m$$

where $\alpha_m + i\beta_m$ are given by (2.38), $\beta_m = 0$, $m \geq 1$, and $\alpha_0 + i\beta_0 = w_0(z)$, and where $u_m + iv_m$ is given by (2.27) forz $m \geq 1$ and $u_0 + iv_0 = w_0(z)$. More generally we shall show that

$$(2.42.1) \qquad \|u_m(x,y) + iv_m(x,y)\|_\mu \leq \|\alpha_m(x,y) + i\beta_m(x,y)\|_\mu .$$

To see this, note that

$$\|u_0(x,y) + iv_0(x,y)\|_\mu = \|\alpha_0(x,y) + i\beta_0(x,y)\|_\mu ,$$

and from (2.25), (2.16.1) and (2.35.1)

$$\|u_1(x,y)+iv_1(x,y)\|_\mu \le L^* r_0 b^0 p_0 c_p + f_0 c_f + (2\pi)^{-1} K^*(\mu) n_0 c_n$$
$$= \alpha_1$$
$$= \|\alpha_1 + i\beta_1\|_\mu .$$

I claim that for all $j \ge 0$,

$$\|u_j + iv_j\|_\mu \le \|\alpha_j + i\beta_j\|_\mu .$$

This follows by induction on j. We know it is true for $j = 0,1$ by the above. Assume it is true for $0 \le j \le m_0$, we shall prove it is true for $j = m_0 + 1$, i.e., assume

$$\|u_j + iv_j\|_\mu \le \|\alpha_j + i\beta_j\|_\mu , \qquad 0 \le j \le m_0,$$

and consider $u_{m_0+1} + iv_{m_0+1}$ for $m_0 \ge 1$. Since by (2.35.n+1), $\beta_{m_0+1} \equiv 0$ and α_{m_0+1} = positive constant for $m_0 \ge 0$, we must show

$$\|u_{m_0+1} + iv_{m_0+1}\|_\mu \le \|\alpha_{m_0+1}\|_\mu = \alpha_{m_0+1} .$$

Note that since u_j and v_j are real,

$$\|u_j\|_\mu, \|v_j\|_\mu \le \|u_j + iv_j\|_\mu \le \|\alpha_j + i\beta_j\|_\mu = \alpha_j$$

for $1 \le j \le m_0$, by the induction hypothesis and (2.35.n+1). But in (2.41), the right-hand side of the equation contains only u_j, v_j, $1 \le j \le m_0$, thus by (2.41) and (2.38), for $m_0 \ge 1$,

$$\|u_{m_0+1} + iv_{m_0+1}\|_\mu \le \sum_{\substack{1 \le j+k+s \le m_0+1 \\ s \ge 1}} U_{jks}\alpha_s^{(j,k)}[\|u_1\|_\mu, \dots, \|v_{m_0}\|_\mu]$$

$$\le \sum U_{jks}\alpha_s^{(j,k)}[\alpha_1, \dots, \alpha_{m_0}]$$

$$= \alpha_{m_0+1} \ ,$$

which gives the desired result, that is (2.47).

The proof of the theorem will be complete if we can show that the series for $\alpha(x,y,\lambda) + i\beta(x,y,\lambda)$ is convergent. To show the convergence of (2.33) and (2.34), we return to (2.29) and let

$$X = \alpha - u_0, \qquad Y = \beta - v_0, \qquad \tilde{\lambda} = \lambda - \lambda_0 \ .$$

Then (2.29) becomes, since $Y \equiv 0$,

$$X = L^* r_0 b^0 p_0 \sum_{\substack{j,k=0 \\ \ell=1}}^{\infty} [a_p X]^j [b_p X]^k [c_p \tilde{\lambda}]^\ell + f_0 \sum_{\substack{j,k=0 \\ \ell=1}}^{\infty} [a_f X]^j [b_f X]^k [c_f \tilde{\lambda}]^\ell + (2\pi)^{-1} K^*(\mu) n_0 \sum_{\substack{j,k=0 \\ \ell=1}}^{\infty} [a_n X]^j [b_n X]^k [c_n \tilde{\lambda}]^\ell \ . \tag{2.43}$$

Let

$$a = \max\{a_p, a_f, a_n\}, \qquad b = \max\{b_p, b_f, b_n\}, \qquad c = \max\{c_p, c_f, c_n\} .$$

Then the series in (2.43) is majorized by

$$\tilde{\lambda} U^* \sum_{j,k,\ell=0}^{\infty} [aX]^j [bX]^k [c\tilde{\lambda}]^\ell \tag{2.44}$$

where

$$U^* = c[L^* r_0 b^0 p_0 + f_0 + (2\pi)^{-1} K^*(\mu) n_0] ,$$

and (2.44) is majorized by

$$\tilde{\lambda} U^* \sum_{\substack{j+k=0 \\ \ell=0}}^{\infty} [(a+b)X]^{j+k} [c\tilde{\lambda}]^\ell .$$

Thus if we can show

$$X = \tilde{\lambda} U^* \frac{1}{1-(a+b)X} \frac{1}{1-c\tilde{\lambda}} \tag{2.45}$$

has a unique solution that is analytic in $\tilde{\lambda}$, we shall have completed the proof. But (2.45) has for its solution, which vanishes at $\tilde{\lambda} = 0$:

$$X = d - \sqrt{d^2 - t(\tilde{\lambda})} \tag{2.46}$$

with

$$d = [2(a+b)]^{-1} ,$$

$$t(\tilde{\lambda}) = \frac{U^*}{a+b} \frac{\tilde{\lambda}}{1-c\tilde{\lambda}}$$

which is an analytic function in $\tilde{\lambda}$ and is the unique solution of (2.45) with $\tilde{\lambda} = 0$, provided $\tilde{\lambda}$ lies within the circle centered at

$(-ec,0)$ of radius $\sqrt{e^2c^2+e}$ where

$$e = [(2d^{-1}U^*)^2 - c^2]^{-1}.$$

This proves (C.1) of the theorem.

For the case when λ is real (2.46) is an analytic function of $\tilde{\lambda}$ for all

$$\lambda_0 - \frac{d}{2U^* - cd} < \lambda < \lambda_0 + \frac{d}{2U^* + dc}$$

which gives (C.2).

To prove (C.3), we note from (2.21), (2.22) and (2.23) that

$$w(z,\lambda) - w_0(z) = \sum_{k=1}^{\infty} \frac{1}{k!} [u_k(x,y) + iv_k(x,y)]\tilde{\lambda}^k ,$$

then by (2.42.1) and (2.35.n+1),

$$\|w(z,\lambda) - w_0(z)\|_\mu \le \sum_{k=1}^{\infty} \frac{1}{k!} |\tilde{\lambda}|^k \alpha_k \qquad (2.47)$$

$$= \tilde{\alpha}(x,y,|\tilde{\lambda}|) - u_0(x,y)$$

where

$$\tilde{\alpha}(x,y,\tilde{\lambda}) = \tilde{\alpha}(x,y,\lambda-\lambda_0) = \alpha(x,y,\lambda)$$

follows by (2.33). Replacing X by

$$\tilde{X} = \tilde{\alpha} - u_0$$

in (2.43) and proceeding as before, we get from (2.46) that

$$\tilde{\alpha}(x,y,|\tilde{\lambda}|) - u_0(x,y) = d - \sqrt{d^2 - t(|\tilde{\lambda}|)}$$

which with the aid of (2.47) gives (C.3). This completes the proof of Theorem 1.

3. Existence, Uniqueness and Analyticity in λ for λ Small

An immediate consequence of Theorem 1 is the following existence and uniqueness theorem.

COROLLARY 11 *Given the non-linear singular integral equation with real or complex parameter* λ:

$$(E_0) \qquad w(z) = B(z) + \lambda \sum_{r=0}^{r_0} b_r(z) L\{P^r[\sigma, u(\xi,\eta), v(\xi,\eta), \lambda]\}$$

$$+ \lambda F[z, u(x,y), v(x,y), \lambda] + \frac{\lambda}{2\pi i}\int_\Gamma \frac{N[\sigma, z, u(\xi,\eta), v(\xi,\eta), \lambda]}{\sigma - z}\, d\sigma$$

where $z = x + iy \in \Gamma$, $\sigma = \xi + i\eta$ *and the integral is understood to be the Cauchy-Principal value. Assume also:*

(H.1), (H.2), *and* (H.4) *the same as in Theorem* 1 *with* $\lambda_0 = 0$ *and* $w_0(z) = B(z)$. *Then*

(C.1) *There exists a unique single-valued solution* $w(z,\lambda)$ *of* (E_0) *which is an analytic function of* λ *on* $\mathcal{B}_\lambda$ *with* $H(\mu,\lambda)$ *coefficients where* $\mathcal{B}_\lambda$ *is the circle with center at* $-ec$ *and radius* $\sqrt{e^2c^2 + e}$ *where the notation is the same as in* (C.1) *of Theorem* 1.

(C.2)

$$\|w(z,\lambda) - B(z)\|_{\mu} \leq d - \sqrt{d^2 - t(|\lambda|)}$$

where the notation of (C.1) *and* (C.3) *of Theorem* 1 *is used.*

Proof: Specialize Theorem 1.

4. *Analytic Continuation with Respect to* λ

We are now in a position to consider the analytic continuation of the solution with respect to the parameter λ.

THEOREM 2. (Continuation of the solution in parameter space) *Given the non-linear singular integral equation with real or complex parameter* λ:

$$w(z) = B(z) + \sum_{r=0}^{r_0} b_r(z) L\{F^r[\sigma, u(\xi,\eta), v(\xi,\eta), \lambda]\} \tag{E}$$

$$+ F[z, u(x,y), v(x,y), \lambda] + \frac{1}{2\pi i} \int_{\Gamma} \frac{N[\sigma, z, u(\xi,\eta), v(\xi,\eta), \lambda]}{\sigma - z}\, d\sigma$$

where $z = x + iy \in \Gamma$ *and* $\sigma = \xi + i\eta \in \Gamma$ *and the integral is understood to be the Cauchy-Principal value. Also assume:*

(H.1) $B(z)$ *and* $b_r(z)$ *are single-valued functions of* z *which are in* $H(\mu)$, $0 < \mu < 1$, *for* $z \in \Gamma$.

(H.2) L *is a complex-valued linear bounded functional defined on the set of complex-valued functions continuous on* Γ.

(H.3) *A priori hypothesis. For* $\mathcal{D}_\lambda$ *and* $\mathcal{D}_w$ *simply-connected open sets in the* λ*-plane (if* λ *real then* $\mathcal{D}_\lambda$ *is open interval) and* $u + iv$*-plane respectively we have: if* $\lambda \in \mathcal{D}_\lambda$ *and if* $w(z,\lambda) \in H(\mu)$ *is* *a solution of* (E), *then*

$$w(z,\lambda) \in \mathcal{D}_w^*(\mu) \quad \textit{for} \quad z \in \Gamma.$$

(H.4) $P^\Gamma[z,u,v,\lambda]$ *and* $F[\ldots]$ *are defined and single-valued on* $\Gamma \times \mathcal{D}_w \times \mathcal{D}_\lambda$ *and* $N[\sigma,z,u,v,\lambda]$ *on* $\Gamma \times \Gamma \times \mathcal{D}_w \times \mathcal{D}_\lambda$ *and all are uniformly analytic functions on* $\mathcal{D}_w^*(\mu) \times \mathcal{D}_\lambda$ *with* $H(\mu;\, p_0,a_p,b_p,c_p)$, $H(\mu;\, f_0,a_f,b_f,c_f)$ *and* $H(\mu,\mu_1;\, n_0,a_n,b_n,c_n)$ $(0 < \mu < \mu_1 < 1)$ *coefficients respectively.*

(H.5) $\lambda_0 \in \mathcal{D}_\lambda$ *and* $w_0(z) = w(z,\lambda_0) = u_0(x,y) + iv_0(x,y) \in H(\mu)$ *with constant* h_0 *is a solution of* (E) *with* $w_0(z) \subset \mathcal{D}_w$ *for* $z \in \Gamma$.

Conclusion. The unique solution of (E) *which is an analytic function on* $\mathcal{D}_{\lambda_0}$ (*see Theorem* 1), *on a neighborhood of* λ_0, *with* $H(\mu,\lambda)$ *coefficients, can be uniquely continued to be a solution of* (E), *which is an analytic function with* $H(\mu,\lambda)$ *coefficients, throughout* $\mathcal{D}_\lambda$.

Proof: Let λ^* be an arbitrary point of $\mathcal{D}_\lambda$. Join λ_0 to λ^* by means of a path lying completely in $\mathcal{D}_\lambda$. Since $\mathcal{D}_\lambda$ is open, the distance r_1 of the path to the exterior of $\mathcal{D}_\lambda$ is > 0.

According to the conclusion (C.1) of Theorem 1 we know that there is a unique solution $w^{(0)}(z,\lambda)$ of (E) which is an analytic function on B_0 with $H(\mu,\lambda)$ coefficients where B_0 is the circle with center at λ_0 of radius $r = \min\{r_1,r_2\}$ where $r_2 = \sqrt{e^2c+e} - ec$ and where the notation is the same as Theorem 1.

Now we proceed exactly as in the circle chain method of complex variables. Viz. choose points $\lambda_0,\lambda_1,\lambda_2,\ldots,\lambda_p = \lambda^*$ on the path joining λ to λ^* for which the distance, along the path, between adjacent points is $< r$; put circles C_j of radius r centered at λ_j, $j = 1,2,\ldots,p$. Let the open disc whose boundary is C_j be denoted by B_j. Due to the uniformity, all circles are of the same size.

According to Theorem 1, we have a unique solution $w^{(0)}(z,\lambda)$ of (E) which is an analytic function in B_0 with $H(\mu,\lambda)$ coefficients. Moreover λ_1 lies in B_0, by construction, thus $w_1(z) = w^{(0)}(z,\lambda_1)$ is defined and is $\in H(\mu)$ and is a solution of (E), and thus by (H.3), $w_1(z) \in \mathcal{D}^*_w(\mu)$ for $z \in \Gamma$. By Theorem 1, with $w_0(z)$ replaced by $w_1(z)$, we conclude that there exists a unique solution $w^{(1)}(z,\lambda)$ of (E) which is an analytic function on B_1 with $H(\mu,\lambda)$ coefficients. The uniformity insures that $w^{(1)}$ is analytic in λ on all of B_1, and by uniqueness of the solution of (E), we have $w^{(1)}(z,\lambda)$ and $w^{(0)}(z,\lambda)$ agree on the overlap in the λ-plane. Continuing in this way we get the solution of (E) at λ^* which is an analytic function with $H(\mu)$ coefficients. This completes the proof since $\mathcal{D}_\lambda$ is simply connected.

In Theorem 2, we knew, because of uniformity, how far we could extend the solution in each analytic continuation and the circles were all the same size. However it is important to consider the case when we do not have uniformity. In this case, the circle into which we extend the solution, depends on the previous solution. We shall restrict ourselves to the case when λ is <u>real</u>.

<u>Notation:</u>

$$A(x_1,x_2) = \max\,\{a_p(x_1,x_2), a_f(x_1,x_2), a_n(x_1,x_2)\}\ , \tag{4.1}$$

$$B(\ldots) = \max\,\{b_p(\ldots), b_f(\ldots), b_n(\ldots)\}\ , \tag{4.2}$$

$$C(\ldots) = \max\,\{c_p(\ldots), c_f(\ldots), c_n(\ldots)\}\ , \tag{4.3}$$

$$D(\ldots) = \{2[A(\ldots) + B(\ldots)]\}^{-1}, \tag{4.4}$$

$$E(\ldots) = \{[2D^{-1}(\ldots)\tilde{U}(\ldots)]^2 - C^2(\ldots)\}^{-1}, \tag{4.5}$$

$$\tilde{U}(\ldots) = C(\ldots)[r_o b^o L^* p^*(\ldots) + f^*(\ldots) + (2\pi)^{-1}K^*(\mu)n^*(\ldots)] \tag{4.6}$$

(where $K^*(\mu)$ is given in Theorem 1 and the corollary to Lemma 2),

$$T_{\bar{\bar{\epsilon}}}(x_1,x_2) = [A(\ldots) + B(\ldots)]^{-1}\,\tilde{U}(\ldots)R_{\bar{\bar{\epsilon}}}(\ldots)[1 - C(\ldots)R_{\bar{\bar{\epsilon}}}(\ldots)]^{-1} \tag{4.7}$$

(where $\bar{\bar{\epsilon}}$ is arbitrary provided $1 > \bar{\bar{\epsilon}} > 0$),

$$R_{\bar{\bar{\epsilon}}}(x_1,x_2) = (1-\bar{\bar{\epsilon}})D(\ldots)\{2\tilde{U}(\ldots) + D(\ldots)C(\ldots)\}^{-1}\ , \tag{4.8}$$

$$H_{\bar{\bar{\epsilon}}}(x_1,x_2) = D(x_1,x_2) - \sqrt{D^2(\ldots) - T_{\bar{\bar{\epsilon}}}(\ldots)}\ , \tag{4.9}$$

$$\bar{\bar{F}}(x_1,x_2) = [H_{\bar{\bar{\epsilon}}}(\ldots), R_{\bar{\bar{\epsilon}}}(\ldots)]\ . \tag{4.10}$$

THEOREM 3. (Continuing continuation) *Given the non-linear singular integral equation* (E) *with real parameter* λ (*see Theorem* 2):

(H.1)* *Same as* (H.1) *of Theorem* 2 *with* $\|b_r(z)\|_\mu \le b^0$.

(H.2)* *Same as* (H.2) *of Theorem* 2 *with* $L^* = \|L\|$.

(H.3)* *A priori hypothesis. Let* $\mathcal{D}_\lambda$ *be an open interval* $(\lambda_0^*, \lambda_0^{**})$ *containing* λ_0 *and let* $\mathcal{D}_w$ *be an open simply-connected set in the* $u+iv$- *plane such that if* $\lambda \in \mathcal{D}_\lambda$, $w(z,\lambda) \in H(\mu)$ *is a solution of* (E), *then*

$$w(z,\lambda) \in \mathcal{D}_w(\mu) \qquad \text{for} \quad z \in \Gamma .$$

(H.4)* *Let* $P^r[z,u,v,\lambda]$ *and* $F[\ldots]$ *be defined and single-valued on* $\Gamma \times \mathcal{D}_w \times \mathcal{D}_\lambda$, $N[t,z,u,v,\lambda]$ *on* $\Gamma \times \Gamma \times \mathcal{D}_w \times \mathcal{D}_\lambda$, *and all are analytic functions on* $\mathcal{D}_w \times \mathcal{D}_\lambda$ *with* $H(\mu)$ *coefficients for* P^r *and* F, *and* $H(\mu,\mu_1)$ $(0 < \mu < \mu_1 < 1)$ *coefficients for* N.
More specifically: if $(u^* + iv^*, \lambda^*) \in \mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$, *then we have expansions for* P^r, F *and* N *of the form* (2.15.1), (2.15.2) *and* (2.15.3) *with* (u_0, v_0, λ) *replaced by* (u^*, v^*, λ^*) *and in place of* (2.16.1) - (2.16.3) *we have*

$$\|p^r_{jk\ell}(\sigma)\|_\mu \le p^*(h^*,\lambda^*)a_p^j(h^*,\lambda^*)b_p^k(\ldots)c_p^\ell(\ldots) , \tag{4.11}$$

$$\|f_{jk\ell}(\sigma)\|_\mu \le f^*(h^*,\lambda^*)a_f^j(\ldots)b_f^k(\ldots)c_f^\ell(\ldots) , \tag{4.12}$$

$$\|n_{jk\ell}(\sigma)\|_{(\mu,\mu_1)} \le n^*(\ldots)a_n^j(\ldots)b_n^k(\ldots)c_n^\ell(\ldots) , \tag{4.13}$$

where if $w^*(z) = u^*(x,y) + iv^*(x,y)$, *then*

$$\|w^*(z)\|_\mu \le h^* .$$

(H.5)* $\lambda_0 \in \mathcal{D}_\lambda$ and

$$w_0(z) = w(z,\lambda_0) = u_0(x,y) + iv_0(x,y) \in H(\mu)$$

(with $\|w_0(z)\|_\mu \le h_0$) is a solution of (E) with $w_0(z) \subset \mathcal{D}_w$ for $z \in \Gamma$.

(H.6)* Let the unique solution (see Theorem 1) of (E) in $H(\mu)$ which is an analytic function of λ, be valid on the open interval $(\lambda_0^1, \lambda_0^2)$ containing λ_0 and assume $(\lambda_0^1, \lambda_0^2) \subset \mathcal{D}_\lambda$.

Conclusion: For $k \ge 1$, let

$$q_{k-1} = q(h_{k-1}, \lambda_{k-1}) \equiv \{R_{\tilde{\epsilon}}[h_{k-1}, \lambda_{k-1}]\}^{-1} R_{\tilde{\epsilon}}[h_k, \lambda_k]$$

where

$$h_k = h_{k-1} + H_{\tilde{\epsilon}}(h_{k-1}, \lambda_{k-1}) ,$$

$$\lambda_k = \lambda_{k-1} + R_{\tilde{\epsilon}}(h_{k-1}, \lambda_{k-1}) .$$

Then $q_{k-1} \ge 0$ and:

Case 1. If $q_k \ge 1$ for $k \ge$ some $K \ge 0$, then we can uniquely continue the solution to the right of λ_0^2 out to $(\lambda_0^1, \lambda_0^{**})$.

Case 2. If $q_k < 1$ for $k \ge$ some $K \ge 1$, then we can continue the solution, after m continuations $m \ge K$, at least to

$$\tilde{\lambda} = \min\{\lambda_0^{**}, \bar{\lambda}\}$$

where

$$\bar{\lambda} = \lambda_0 + R_{\bar{\varepsilon}}(h_0,\lambda_0)\left\{1 + \sum_{k=1}^{K} \prod_{j=0}^{k-1} q_j + (1 - Q_{K_m}^{m})(1 - Q_{K_m})^{-1} \prod_{j=0}^{K} q_j\right\}$$

with

$$Q_{K_m} = \min\{q_K, q_{K+1}, \ldots, q_m\}$$

and $1 > \bar{\varepsilon} > 0$ *is as small as we like.*

Case 2.1. If, for m *arbitrary but fixed,*

$$0 < q = \min_{m \geq k \geq 1} \{q_k\} < 1 \ ,$$

then we can tontinue the solution, after m *continuations, at least to the right of* λ_0 *as far as*

$$\lambda = \min\{\lambda_0^{**}, \bar{\bar{\lambda}}\}$$

where

$$\bar{\bar{\lambda}} = \lambda_0 + R_{\bar{\varepsilon}}(h_0,\lambda_0)\,\frac{1 - q^{m+1}}{1 - q} \ . \tag{4.14}$$

In particular, if $0 < q = \inf_{k \geq 1}\{q_k\} < 1$ *and*

$$q > 1 - \frac{R_{\bar{\varepsilon}}(h_0,\lambda_0)}{\lambda_0^{**} - \lambda_0} \ , \tag{4.15}$$

then there exists an m_0 *such that after* m_0 *continuations we reach*

λ_0^{**}, _i.e., we can continue the solution to the right end point of_ $\mathcal{D}_\lambda$, _i.e., out to_ $(\lambda_0^1, \lambda_0^{**})$. m_0 _can be taken to be such that:_

$$q^{m_0+1} \le \frac{R_{\bar{\varepsilon}}(h_0,\lambda_0) - (1-q)(\lambda_0^{**}-\lambda_0)}{R_{\bar{\varepsilon}}(h_0,\lambda_0) + (1-q)\lambda_0} .$$

Proof: According to (C.2) of Theorem 1, we have a unique solution $w(z,\lambda)$ in $H(\mu)$ for

$$\begin{aligned} \lambda = \lambda_1 &= \lambda_0 + (1-\bar{\varepsilon})(\lambda_0^2 - \lambda_0) \\ &= \lambda_0 + R_{\bar{\varepsilon}}(h_0,\lambda_0) \end{aligned} \tag{4.16}$$

where $R_{\bar{\varepsilon}}$ is defined by (4.8).

Moreover by (C.3) of Theorem 1, we have

$$\begin{aligned} \|w(z,\lambda_1)\|_\mu &\le \|w_0(z)\|_\mu + \|w_0(z) - w(z,\lambda_1)\|_\mu \\ &\le h_0 + H_{\bar{\varepsilon}}(h_0,\lambda_0) \end{aligned}$$

where $H_{\bar{\varepsilon}}$ is given by (4.9). Let

$$h_1 \equiv h_0 + H_{\bar{\varepsilon}}(h_0,\lambda_0) .$$

If we let

$$w_1(z) = w(z,\lambda_1) ,$$

then we can apply Theorem 1 again with λ_0 replaced by λ_1 and h_0 by h_1. Continuing in this way, we get

$$h_{k+1} = h_k + H_{\bar{\varepsilon}}(h_k,\lambda_k) ,$$

$$\lambda_{k+1} = \lambda_k + R_{\bar{\varepsilon}}(h_k,\lambda_k) ,$$

where $H_{\bar{\epsilon}}$ and $R_{\bar{\epsilon}}$ are given respectively by (4.9) and (4.8).

Thus

$$\begin{aligned}\lambda_{k+1} &= \lambda_0 + (\lambda_1 - \lambda_0) + (\lambda_2 - \lambda_1) + \cdots + (\lambda_{k+1} - \lambda_k) \\ &= \lambda_0 + R_{\bar{\epsilon}}(h_0, \lambda_0) + R_{\bar{\epsilon}}(h_1, \lambda_1) + \cdots + R_{\bar{\epsilon}}(h_k, \lambda_k)\end{aligned}$$

and

$$q_{k-1} \equiv \frac{R_{\bar{\epsilon}}[h_k, \lambda_k]}{R_{\bar{\epsilon}}[h_{k-1}, \lambda_{k-1}]} = \frac{\lambda_{k+1} - \lambda_k}{\lambda_k - \lambda_{k-1}} .$$

Note that

(4.17) $$\lambda_{k+1} = \lambda_0 + (\lambda_1 - \lambda_0)(1 + q_0 + q_0 q_1 + \cdots + q_0 q_1 \cdots q_{k-1})$$

from which Case 1 follows immediately.

As for Case 2, note that for $m \geq K$,

$$\begin{aligned}\lambda_{m+1} &\geq \lambda_0 + (\lambda_1 - \lambda_0)\left\{1 + \sum_{k=1}^{K} \prod_{j=0}^{k-1} q_j + \prod_{j=0}^{K} q_j \sum_{\ell=0}^{m-1} Q_{K_m}^{\ell}\right\} \\ &= \lambda_0 + (\lambda_1 - \lambda_0)\left\{1 + \sum_{k=1}^{K} \prod_{j=0}^{k-1} q_j + [(1 - Q_{K_m}^{m})/(1 - Q_{K_m})] \prod_{j=0}^{K} q_j\right\} ,\end{aligned}$$

which is the result of Case 2.

(4.14) of Case 2.1 follows immediately from (4.17) if we replace q_j by q and use (4.16).

If condition (4.15) holds, then we get

$$1 - q \equiv \frac{R_{\epsilon}(h_o, \lambda_o)}{\lambda_o^{**} - \lambda_o + \delta} \tag{4.18}$$

where $\delta > 0$ is some fixed number. Thus

$$\begin{aligned}
\bar{\lambda} &= \lambda_0 + (1 - q^{m+1})(\lambda_0^{**} - \lambda_0 + \delta) \\
&= \lambda_0 q^{m+1} + (1 - q^{m+1})(\lambda_0^{**} + \delta) \\
&\geq (1 - q^{m+1})(\lambda_0^{**} + \delta) \\
&\geq \lambda_0^{**}
\end{aligned}$$

provided m_0 is such that $q^{m_0+1} \leq \frac{\delta}{\delta + \lambda_o^{**}}$ which is always possible since $q < 1$. Thus Case 2.1 is proved. To get an estimate for m_0 we simply use the expression for δ given in (4.18).

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Footnotes

* The author wishes to greatfully acknowledge support for this research by the National Aeronautics Space Administration, NASA Grant NGR 05-010-008.

** By this is meant: if $f(\sigma)$ and $g(\sigma)$ are functions continuous on Γ then $L[a(\lambda)f(\sigma) + b(\lambda)g(\sigma)] = a(\lambda)L[f(\sigma)] + b(\lambda)L[g(\sigma)]$ and $|L[f(\sigma)]| \le L_1^* \max_{\sigma \in \Gamma} |f(\sigma)|$, where L^* is independent of f.

Let $L^* = \|L\| = \sup \dfrac{|L[f]|}{\max |f(0)|}$.